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A modified weakest-link model for describing strength variability of Kevlar aramid fibres

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The mathematical description of a weakest-link model for fibre strength was developed in which there is no assumption regarding the test length with respect to the link length. The filament strength as a function of test length of three commercial Kevlar aramid yarns was compared with the model predictions. The model was found to fit the data well.

1. Introduction

The variability in tensile strength of fibres used for reinforcing composite structures is an important factor in determining the tensile strength of the final structure relative to the actual average strength of the individual filaments. Much theoretical analysis has been carried out in an effort to understand the tensile failure processes of fibre arrays and to predict the strength of an array based on its filament properties [1–7]. One aspect of this effort is the characterization of the magnitude and nature of the filament strength variability.

Kevlar* aramid is a high modulus and strength fibre produced by E. I. Du Pont De Nemours & Co. Inc. It is manufactured by spinning a liquid crystalline solution of poly *p*-phenylene terephthamide (PPD-T) [8]. This technique permits alignment of the rigid PPD-T molecules along the fibre axis. The high degree of molecular orientation is the primary fibre structural property responsible for the high modulus and strength [9, 10]. Three types of Kevlar are available. Kevlar and Kevlar 29 are high strength and modulus products which are used mainly in ballistics, ropes, cables and rubber reinforcing applications. Kevlar 49 has a higher modulus than the others [11]. Kevlar 49 is used primarily in composites for aerospace applications.

A fibre strength model commonly employed to describe mean strength and variability and their dependence on test length is the weakest link theory [12]. This model assumes that the length of fibre tested can, with respect to strength, be described as a series of a large number of randomly assembled links of which the strengths are independent, identically distributed, random variables with a common cumulative distribution function. The cumulative distribution of the fibre strengths is then given by

$$F_n(s) = 1 - [1 - F(s)]^n \qquad (1)$$

where $F(s)$ is the common cumulative distribution of the link strengths, s is the strength and n is the number of links needed to describe the fibre. In somewhat different terms, $1\text{-}F_n(s)$ is the probability that a given fibre sample containing n links will be unbroken at an applied stress of s.

The Weibull distribution [13] is often used to describe the strength distribution of high modulus and strength fibres [1]. The Weibull cumulative distribution function is

$$F(s) = 1 - \exp[-(s/a)^b] \quad s >= 0 \quad a, b > 0 \qquad (2a)$$

$$\bar{s} = a\Gamma(1 + 1/b) \qquad (2b)$$

$$cv = \frac{[\Gamma(1 + 2/b) - \Gamma^2(1 + 1/b)]^{1/2}}{\Gamma(1 + 1/b)} \qquad (2c)$$

where $\bar{s}$ is the mean, cv is the coefficient of variation, and Γ is the gamma function.

Combination of the Weibull distribution (Equation 2a) with the weakest link relationship results in the following expression

$$F_n(s) = 1 - \exp[-n(s/a)^b] \qquad (3a)$$

Further, if λ is the length of the links necessary to describe the fibre and L is the actual length of the specimen, then, when $L \gg \lambda$, L/λ very close approximates n, and

$$\bar{s} = \frac{\lambda^{1/b} a\Gamma(1 + 1/b)}{L^{1/b}} \qquad (3b)$$

The expression for the coefficient of variation is unaffected by the weakest link relationship. It is constant as L varies and is given by Equation 2c. It is also interesting to note that Equation 3a with $n = L/\lambda$ can also be derived from a Poisson model in which the mean number of defects per unit length with strength $= < s$ is $(s/a)^b/\lambda$. The impossibility of distinguishing between this special case of the Poisson model, which deals with dimensionless defects, and the classical weakest-link model, which considers links of finite length, is due to the asssumption that $L \gg \lambda$.

Equation 3b predicts that the logarithm of the mean fibre strength will be a linear function of the logarithm of the specimen length with a slope of $-1/b$. It has been

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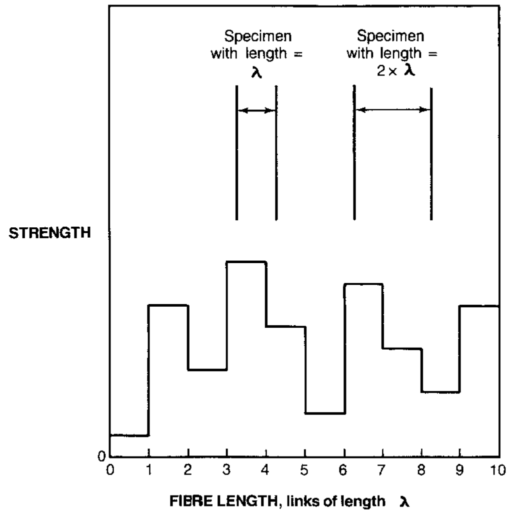

Figure 1 Strength variation along length of the model fibre.

observed by this researcher and others [14] that this expression does not adequately describe the behaviour of the filament strength of Kevlar as a function of specimen length. Generally, mean strength will tend to plateau to a constant value at small specimen lengths, whereas Equation 3b predicts a constant increase.

2. Model description

The modified model is based on the supposition that the physical model used for the classical weakest-link theory is correct for Kevlar, but that the requirement that the test length be much greater than the link length is not met. Therefore, the relationship for the cumulative distribution function (Equation 1) is not valid and must be reformulated.

Consider a fibre of which the strength can be adequately described by a series of independent links of length λ. The strength along a segment of this fibre could be represented by Fig. 1. Assume that we know the value of λ and that our initial testing on this fibre will be done at test lengths, L_i, which are integer multiples of the link length. That is,

$$L_i = i\lambda, \qquad i = 1, 2, 3, \ldots \tag{4}$$

The key point of this model immediately becomes obvious as we consider the first series of tensile tests at, for example, $L_i = \lambda\,(i = 1)$. In the random process of specimen selection, it is only in the very unlikely event when we choose that the specimen ends exactly at the boundaries of a link that the test specimen contains only one full link. At all other times, it will contain segments of two links. Because within links the strength is assumed to be constant, the amount of the link included in the specimen is not relevant with respect to influencing the strength. Thus, the strength at test length $L_i = \lambda$ will follow very closely the cumulative distribution function

$$F_1(s) = 1 - [1 - F(s)]^2 \tag{5}$$

At test lengths which are higher integer multiples, i, of the link length, the specimens will essentially always contain $i - 1$ full links plus segments of two additional links. The net result is that the specimen strengths will be based on a "weakest-link" distribution involving $i + 1$ links.

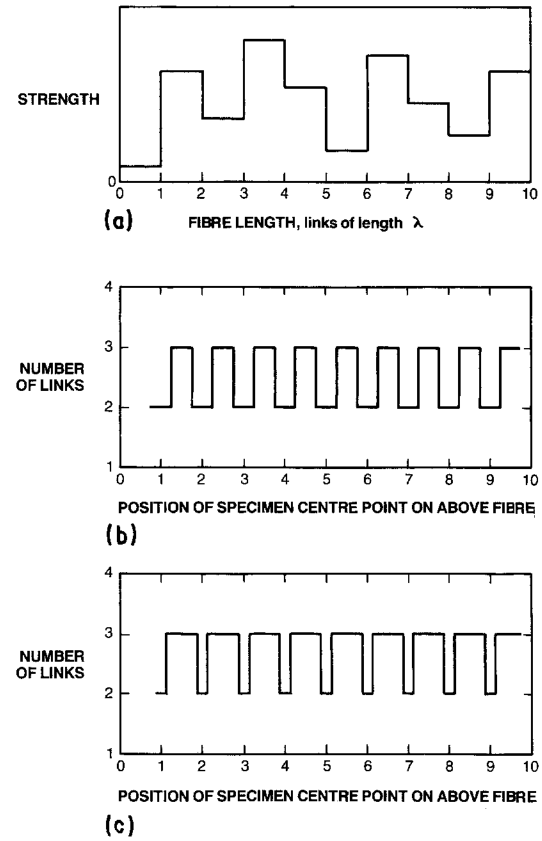

Figure 2 Variation in the number of links represented in a specimen as a function of position of specimen centre point. (a) Model fibre from which specimens are selected; (b) $L/\lambda = 1.5$, mean number of links per specimen = 2.5; (c) $L/\lambda = 1.75$, mean number of links per specimen = 2.75.

$$F_i(s) = 1 - [1 - F(s)]^{i+1} \tag{6}$$

or

$$F_i(s) = 1 - [1 - F(s)]^{L_i/\lambda+1} \tag{7}$$

The practical usefulness of the derived cumulative distribution function relationship (Equation 6) depends on its being generalized to represent situations where the test length is not restricted to an integer multiple of the link length. In these cases, the continuous test length variable will be designated L.

For a given value of L, the number of links represented in randomly chosen specimens will be either the largest integer in $L/\lambda + 1$ or the largest integer in $L/\lambda + 2$. The number of links represented will depend on the position of the specimen with respect to the links. Fig. 2 shows how the number of links represented varies as a function of the position of the specimen centre point on the model fibre for $L/\lambda = 1.5$ and $L/\lambda = 1.75$. Examination of these functions shows that for a large number of randomly chosen specimens, the fraction of specimens with the larger number of links represented will be the fractional portion of L/λ. The remaining fraction of specimens will have the lower number of links represented. For example, if $L/\lambda = 1.75$, then 25% of the randomly chosen specimens will have two links represented and 75% will have three links represented. The mean number of links represented in these specimens will be, for this example, 2.75 or, in general, $L/\lambda + 1$.

The more general form of the specimen cumulative distribution function based on the above discussion, which allows for noninteger values of L/λ, is

$$F_L(s) = 1 - [1 - \text{frac}(L/\lambda)][1 - F(s)]^{\text{int}(L/\lambda)+1} - [\text{frac}(L/\lambda)][1 - F(s)]^{\text{int}(L/\lambda)+2} \quad (8)$$

where $\text{int}(L/\lambda)$ is the largest integer $\leqslant L/\lambda$, and $\text{frac}(L/\lambda)$ is the fractional portion of L/λ. This relationship simplifies to Equation 7 when L/λ is an integer.

An attractive approximation to the generalized relationship (Equation 8) is

$$F_L(s) = 1 - [1 - F(s)]^{L/\lambda+1} \quad (9)$$

This relationship is Equation 7 in which the restriction that the specimen length be an integer multiple of λ has been relaxed. The strength means and coefficients of variation have been numerically calculated for a wide range of L/λ values (0.1 to 100) and for a Weibull distribution of link strengths with *cv*'s of 5% to 25% using both exact and approximate relationships. The maximum difference in predicted mean strengths was less than 1%. The maximum difference for strength coefficient of variation is less than 5% (relative). The error introduced by the approximate equation is negligible with respect to the uncertainty associated with available data.

In the limits of very small and very large specimen lengths, the derived relationships (Equations 8 and 9) are physically realistic. As L/λ becomes very large

$$1 - [1 - F(s)]^{L/\lambda+1} \simeq 1 - [1 - F(s)]^{L/\lambda} \quad (10)$$

which is a form of the classical "weakest-link" relationship. As L/λ becomes very small

$$1 - [1 - F(s)]^{L/\lambda+1} \simeq F(s) \quad (11)$$

which states that, in this situation, the specimen cumulative distribution function becomes that of the links. This is the correct result for a continuous determination of the fibre strength which is what $L = 0$ implies.

Substitution of the Weibull cumulative distribution function (Equation 2) into the generalized expressions (Equations 8 and 9) results in the following relationship,

$$F_L(s) = 1 - [1 - \text{frac}(L/\lambda)] \exp[(\text{int}(L/\lambda) + 1)(s/a)^b] - [\text{frac}(L/\lambda)] \exp[(\text{int}(L/\lambda) + 2)(s/a)^b] \quad (12)$$

and for the approximate form,

$$F_L(s) = 1 - \exp[-(L/\lambda + 1)(s/a)^b] \quad (13a)$$

This expression is particularly convenient because the resulting mean strength is

$$\bar{s} = \frac{a\Gamma(1 + 1/b)}{(L/\lambda + 1)^{1/b}} \quad (13b)$$

The coefficient of variation of the strength is

$$cv = \frac{[\Gamma(1 + 2/b) - \Gamma^2(1 + 1/b)]^{1/2}}{\Gamma(1 + 1/b)} \quad (13c)$$

which is a function only of the Weibull parameter, b and is identical to Equation 2c. Unlike Equation 3b, Equation 13b is not derivable from a Poisson model.

3. Experimental methods

The strength (breaking load) of Kevlar filaments is determined by a technique which has been developed in our laboratory especially for this fibre. All filaments are mounted on paper tabs with an amine catalysed cyanoacrylate adhesive. We have found this mounting technique to be quite satisfactory in terms of reproducibility, convenience and mounting speed. Tensile testing is carried out on an "Instron" Model 1122 tester equipped with 500 gram capacity pneumatic

TABLE I Filament breaking load data: item A

	Filament							
	1	2	3	4	5	6	7	8
Test length = 0.18 cm								
Mean (dN)	4.71	4.33	4.56	4.88	4.72	4.46	4.63	5.09
Standard deviation	0.36	0.65	0.36	0.31	0.34	0.36	0.12	0.18
$n = 10$								
Test length = 1.0 cm								
Mean (dN)	4.35	4.47	4.97	4.31	4.47	4.49	4.26	5.63
Standard deviation	0.28	0.42	0.43	0.30	0.25	0.39	0.32	0.20
$n = 10$								
Test length = 2.5 cm								
Mean (dN)	4.25	4.60	3.9	4.88	4.00	4.40	4.76	4.44
Standard deviation	0.31	0.13	0.12	0.14	0.14	0.08	0.26	0.27
$n = 7$								
Test length = 5.1 cm								
Mean (dN)	3.75	3.86	4.86	4.70	3.65	3.91	4.20	4.14
Standard deviation	0.41	0.18	0.42	0.32	0.42	0.21	0.14	0.22
$N = 7$								
Test length = 25.4 cm								
Mean (dN)	3.93	3.37	3.95	3.90	3.73	3.31	4.29	4.24
Standard deviation	0.21	0.76	0.22	0.57	0.30	0.65	0.47	0.23
$n = 7$								

Each mean and standard deviation value represents a different filament. Total of 40 filaments tested.

TABLE II Filament breaking load data: item B

	Filament							
	1	2	3	4	5	6	7	8
Test length = 0.18 cm								
Mean (dN)	4.42	4.12	4.74	4.48	5.47	4.89	5.06	4.84
Standard deviation	0.45	0.45	0.56	0.56	0.38	0.52	0.28	0.44
$n = 10$								
Test length = 1.0 cm								
Mean (dN)	3.88	4.31	6.11	4.75	4.91	4.22	4.80	4.45
Standard deviation	0.29	0.31	0.51	0.40	0.66	0.26	0.59	0.26
$n = 10$								
Test length = 2.5 cm								
Mean (dN)	3.47	4.52	3.95	4.91	3.33	4.43	3.96	4.58
Standard deviation	0.28	0.81	0.42	0.27	0.41	0.38	0.74	0.33
$n = 7$								
Test length = 5.1 cm								
Mean (dN)	5.57	3.94	5.22	4.06	3.55	4.71	4.09	4.02
Standard deviation	0.64	0.40	0.42	0.83	0.40	0.27	0.45	0.56
$N = 7$								
Test length = 25.4 cm								
Mean (dN)	4.34	3.30	2.84	3.35	3.46	34.76	3.84	3.33
Standard deviation	0.67	0.59	0.77	0.53	1.32	1.38	0.76	0.52
$n = 7$								

Each mean and standard deviation value represents a different filament. Total of 40 filaments tested.

grips. The elongation rate was approximately 20% min^{-1}. Testing is done at approximately 70° F and 55 r.h. This we believe presents no special problems because of the relative intensity of Kevlar to any small temperature and humidity changes which may have occurred.

The filament strength of three commercial samples of Kevlar (designated A, B and C) was characterized at five test lengths (0.18, 1.0, 2.5, 5.1 and 25.4 cm). For each test length, eight filaments were randomly sampled from the yarn bundle and multiple breaks done along the length of the filaments. For the 0.18 and 1.0 cm test length, ten breaks were done on each of eight filaments. For the 2.5, 5.1 and 25.4 cm test length, seven breaks were done on each of eight filaments. In the data summary (Tables I, II and III), each mean and standard deviation value represents a different filament (total of 40 filaments sampled).

4. Discussion

The model deals only with a distribution of link strengths along a single hypothetical filament. It ignores any differences in the distribution between filaments and, therefore, is strictly applicable of data

TABLE III Filament breaking load data: item C

	Filament							
	1	2	3	4	5	6	7	8
Test length = 0.18 cm								
Mean (dN)	4.98	4.57	4.86	4.75	4.85	4.95	4.77	4.96
Standard deviation	0.24	0.62	0.57	0.44	0.30	0.37	0.19	0.33
$n = 10$								
Test length = 1.0 cm								
Mean (dN)	4.84	4.66	4.93	5.07	4.52	4.75	4.90	4.71
Standard deviation	0.17	0.28	0.26	0.39	0.36	0.10	0.35	0.31
$n = 10$								
Test length = 2.5 cm								
Mean (dN)	4.78	4.75	4.65	4.88	4.86	4.71	4.60	4.47
Standard deviation	0.18	0.19	0.38	0.37	0.39	0.58	0.20	0.49
$n = 7$								
Test length = 5.1 cm								
Mean (dN)	4.10	4.31	4.43	3.98	4.62	4.43	4.70	4.53
Standard deviation	0.32	0.57	0.31	0.45	0.36	0.66	0.25	0.39
$N = 7$								
Test length = 25.4 cm								
Mean (dN)	4.06	3.93	3.41	3.50	3.62	3.75	4.14	4.00
Standard deviation	0.31	0.73	0.83	0.49	0.40	0.77	0.62	0.20
$n = 7$								

Each mean and standard deviation value represents a different filament. Total of 40 filaments tested.

TABLE IV Filament breaking load summary: item A

Test length (cm)	Breaking load (dN)	Total *CV* (%)	Within-filament *CV* (%)*
0.18	4.67	9.0	7.9
1	4.62	11.7	7.2
2.5	4.40	8.5	4.5
5.1	4.13	12.1	7.5
25.4	3.85	14.5	12.2

*From joint estimate [16].

TABLE VI Filament breaking load summary: item C

Test length (cm)	Breaking load (dN)	Total *CV* (%)	Within-filament *CV* (%)*
0.18	4.84	8.5	8.4
1	4.80	6.7	6.1
2.5	4.71	8.1	7.9
5.1	4.39	10.7	9.9
25.4	3.81	16.0	15.3

*From joint estimate [16].

collected from either a sample in which all filaments have the same distributions of strengths along their length or a sample of a single filament which has uniform distribution of strengths along its length. In practice, neither of these situations is likely or very practical and, therefore, sources of overall variability within a sample must be recognized and only those components pertinent to the model considered.

Only the within-filament component of strength variability should be used when applying the filament strength against test length data presented here to the model. A distribution in mean strength between filaments (the between filament component of strength variability) will contribute to the overall strength coefficient of variation in the yarn sample, but will not impact the response of average strength of coefficient of variation to test length. This fact is clear if one considers a yarn sample composed of perfectly uniform filaments of different strengths. For this hypothetical sample, the measured mean and coefficient of variation would be independent of test length.

A joint estimate of the within-filament strength coefficient of variation at each test length was made (Tables IV, V and VI) [15]. If the within-filament variances constitute a homogeneous population, then this value is an estimate of the common within filament coefficient of variation and the conditions of the model are well satisfied. If the variances are not homogeneous, the value is an estimate of the average within-filament strength coefficient of variation. This is a departure from the assumptions of the model and, depending on the degree of nonhomogeneity, could cause significant differences between the predicted and observed response of strength to test length. Bartlett's chi square statistic indicates that, in most cases, the null hypothesis of variance equivalency can be rejected with a reasonable degree of confidence. The confidence of this rejection, however, may be erroneously high because of the assumption of normality made in the chi square statistic test. Tesing of the variance homogeneity using the gamma plotting technique [16], which is less sensitive to deviations from normality than is the chi square statistic, does not support the rejection of the null hypothesis of variance equivalency. Therefore, it is valid to assume that the jointly estimated coefficient of variation is an estimate of the common within filament coefficient of variation.

The estimated within-filament strength coefficients of variation are, for all three items, reasonably constant for the test lengths of 5.1 cm and less (Figs 3, 4 and 5). For these test lengths, the data are consistent with the model and a Weibull distribution of link strengths (Equation 13c). For all items, the 25.4 cm test length coefficient variation is significantly higher than others. This indicates a breakdown of one or more of the assumptions made to arrive at Equation 13c. Three possible causes for this increase in coefficients of variation are nonhomogeneity of the within-filament link strength distributions from filament to filament, a deviation of the link strength distribution from a Weibull in the low strength tail region or the influence of particulates which could manifest also as a deviation from a Weibull distribution.

The expressions for mean strength for both the classical (Equation 3b) and the modified (Equation 13b) models were fitted to the overall mean breaking load against test length data for each item using a least squares technique. This was done by first estimating the overall coefficient of variation of link strengths from the 5.1 cm and lower test length data of each item (Table VII). With the appropriate value of the Weibull

TABLE V Filament breaking load summary: item B

Test length (cm)	Breaking load (dN)	Total *CV* (%)	Within-filament *CV* (%)*
0.18	4.75	12.4	9.8
1	4.68	16.1	9.3
2.5	4.39	17.0	11.9
5.1	4.15	18.8	11.9
25.4	3.48	26.5	25.2

*From joint estimate [16].

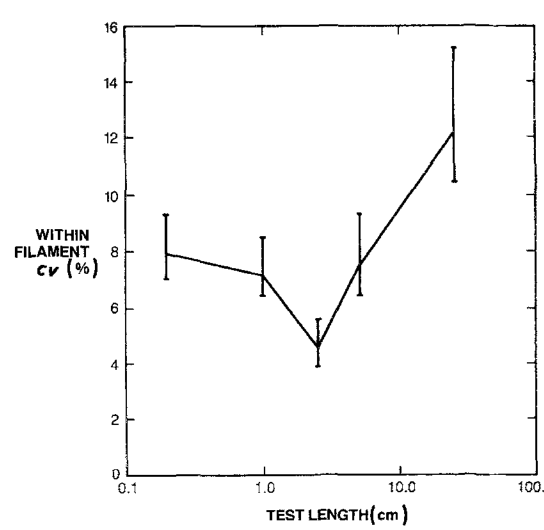

Figure 3 Within-filament breaking load coefficient of variation as a function of test length, item A. Error bars are 95% confidence limits.

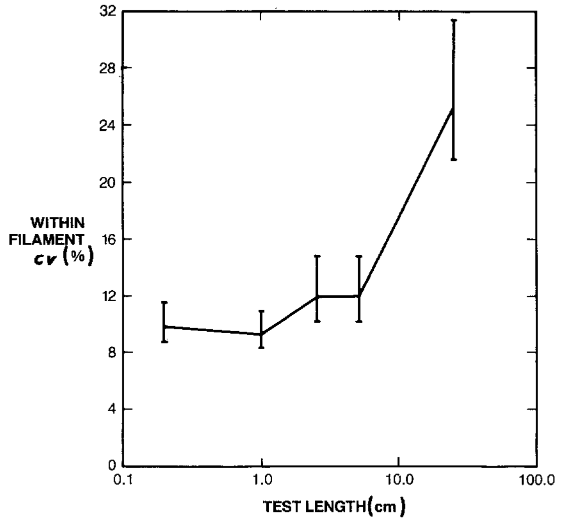

Figure 4 Within-filament breaking load coefficient of variation as a function of test length, item B. Error bars are 95% confidence limits.

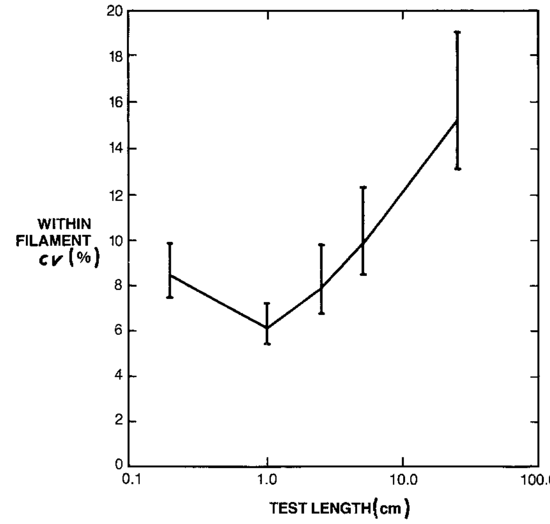

Figure 5 Within-filament breaking load coefficient of variation as a function of test length, item C. Error bars are 95% confidence limits.

b parameter (Equation 13c), the expressions were then fitted to determine the best values of the adjustable parameters. For the modified model (Equation 13b), best values of both the Weibull parameter, a, and the link length, λ, were determined. For the classical model (Equation 3b), in which the link length λ is of no consequence (a result of the assumption that $L \gg \lambda$), a and λ are not resolvable and a best value of $a\lambda^{(1/b)}$ was determined.

The response of the measured overall filament strength to test length is described well by the modified model (Figs 6, 7 and 8). The most significant feature is the levelling off of the mean strength at lower test lengths, which is consistent with the data. This corresponds to the situation in which the test length is comparable to the model link length. At test lengths much smaller than the link length, the measured mean strength will be independent of the test length and will equal to the mean link strength. The classical model predicts that the measured mean strength continues to increase at low test lengths, which is clearly not the case. Whereas the fit of the modified model is good, it is not able to completely describe the data within the 95% confidence limits. This is likely due to deviations of the distribution of link strengths from the assumed Weibull distribution. Further work is now underway to determine the distribution of link strength by computation methods which do not place any restrictions on its shape. It is anticipated that this will improve the fit to the data significantly.

5. Conclusions

A weakest-link model for fibre strength, in which the link length is a parameter, can be suitably formulated mathematically without any conditions as to the size of test length with respect to the link length.

This model describes very well the filament strength mean and coefficient of variation response to test length of three commercial Kevlar aramid yarn samples. The significant increase in the coefficient of variation at the longest test length (25.4 cm) is most likely to be a result of a deviation of the link strength distribution from the Weibull in the low strength tail portion which becomes very influential in determining the observed distribution of strengths when test length becomes substantially larger than the link length.

The fit of the model to the data indicates that, with respect to within-filament strength variations, the filaments of these samples can be considered to be a series of approximately 1 cm length uniform strength links of which the strengths are identically distributed random variables with a common Weibull distribution. A likely physical property of the filaments which would result in within-filament strength variability of this nature is denier variation. Additional investigation

TABLE VII Estimated overall within filament *CV* and Weibull parameter 'B'

Item	*CV* (%)	'B'
A	6.9	18
B	10.8	11
C	8.2	14.5

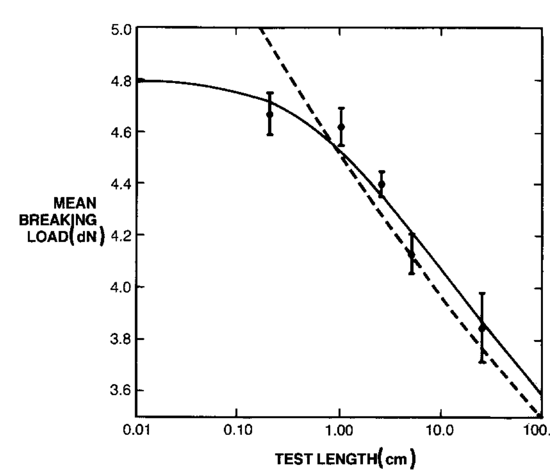

Figure 6 Mean filament breaking load as a function of test length, item A. Error bars are 95% confidence limits. (●) Table IV; (——) Modified weakest link: Equation 13b with a = 5.39 dN, λ = 0.54 cm, b = 18; (– – –) Classical weakest link: Equation 3b with b = 18.

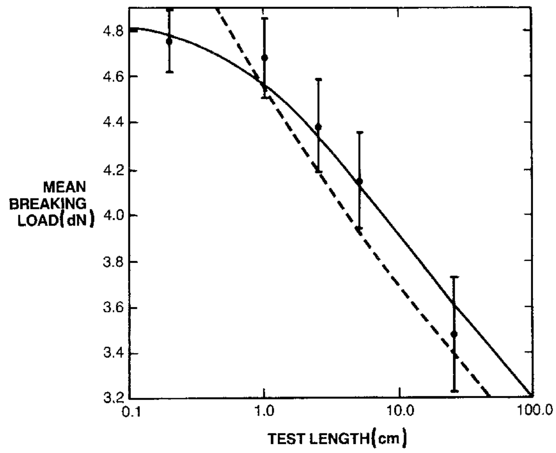

Figure 7 Mean filament breaking load as a function of test length, item B. Error bars are 95% confidence limits. (●) Table V; (——) Modified weakest link: Equation 13b with $a = 5.07$ dN, $\lambda = 1.02$ cm, $b = 11$; (– – –) Classical weakest link: Equation 3b with $b = 11$.

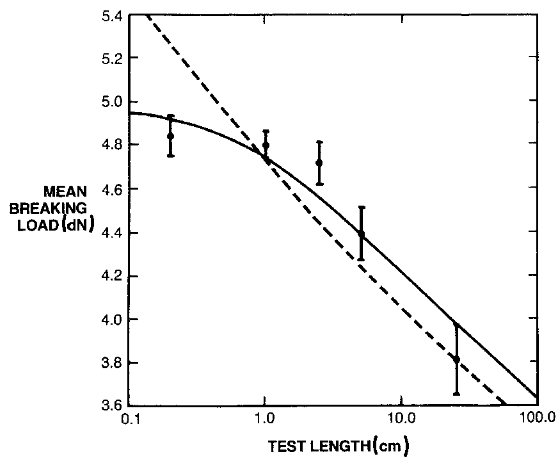

Figure 8 Mean filament breaking load as a function of test length, item C. Error bars are 95% confidence limits. (●) Table VI; (——) Modified weakest link: Equation 13b with $a = 5.14$ dN, $\lambda = 0.88$ cm, $b = 15$; (– – –) Classical weakest link: Equation 3b with $b = 15$.

as to the relationship between the strength and denier variation along Kevlar filaments is now underway.

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